Surface Integrals

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$$\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$$

$\text{curl}(\vec{F}) = 0$ where $\vec{F}$ is defined (ie, on $\mathbb{R}^2 \setminus \{0,0\}$)

" $\vec{F}$ is irrotational" (ie, curl is 0)

conservative $\Rightarrow$ irrotational

irrotational $\Rightarrow$ conservative on a simply-connected domain

But $\mathbb{R}^2 \setminus \{(0,0)\}$ is not SC

Fact every irrotational vector field is locally conservative

ie, say $\vec{F}$ is irrotational on a domain D (ie, open in $\mathbb{R}^2$)

now $\vec{F}$ might not have a potential on D, but $\forall P \in D$, $\exists$ an open neighborhood containing P and contained in D on which $\vec{F}$ is conservative

ie, $\exists \varepsilon > 0$ s.t. $\vec{F}$ has a potential on $D_P(\varepsilon)$

but there might not be a single potential defined on all of D

θ is not a well-defined continuous fcn of $\mathbb{R}^2 \setminus \{(0,0)\}$

eg $\theta((1,1)) = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \quad \frac{\pi}{4} + 2\pi k$ for $k \in \mathbb{Z}$

Usually we choose $\theta \in [0, 2\pi)$

$\Rightarrow \theta$ not continuous on positive x-axis

bc $\lim_{\varepsilon \to 0^+} \theta((1,\varepsilon)) = 0$

$\lim_{\varepsilon \to 0^-} \theta((1,\varepsilon)) = 2\pi$

In a sense $\theta((1,0)) = 0 \overset{?}{=} 2\pi$ ("multivalued fcn")

Recall FTC for line integrals

If $\vec{F} = \nabla F$, C is a curve from P to Q, then $\int_C \vec{F} \cdot d\vec{r} = F(Q) - F(P)$

Now take $F = \theta$, $\vec{F}$ as above

$$\int_{C_{(0,0)}(1)} \vec{F} \cdot d\vec{r} = \underset{2\pi}{\underset{\shortparallel}{F(1,0)}} - \underset{0}{\underset{\shortparallel}{F(1,0)}}$$

$$= 2\pi$$

Idea when you go around in a circle, you end up somewhere different (eg $\rightarrow$ a parking garage)

Notice θ is defined and continuous locally on $\mathbb{R}^2 \setminus \{(0,0)\}$

different (eg → a parking garage)

<u>Notice</u> θ is defined and continuous locally on $\mathbb{R}^2 \setminus \{(0,0)\}$

<u>eg</u> Define $\theta \in [-\pi, \pi)$

Then θ is cont on the positive x-axis but not on the negative x-axis

<u>Winding #</u>

(ie, how to determine whether $P \in \text{int}(C)$.)

Let C be a simple closed curve.

Let $P \in \mathbb{R}^2 \setminus C$. $P := (x_0, y_0)$

<u>Consider</u>

$$\int_C \vec{F}_P \cdot d\vec{r}$$

for $\vec{F} = \dfrac{(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}\hat{\imath} + \dfrac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\hat{\jmath}$

Then this integral is 2π if $P \in \text{int}(C)$

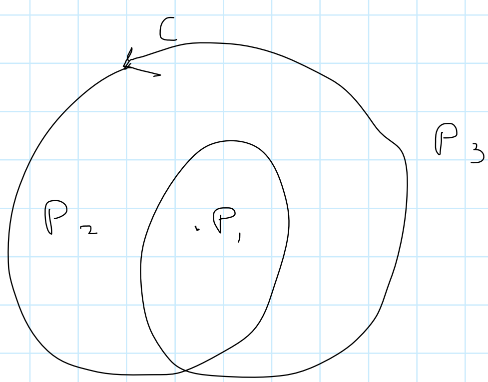
0 $P \in \text{ext}(C)$

<u>in fact</u>

$$\frac{\int_C \vec{F}_P \cdot d\vec{r}}{2\pi}$$ is the # of times that C goes around P.

↳ true even if C not simple closed

<u>eg</u>



winding # of C around

$P_1 = 2$

$P_2 = 1$

$P_3 = 0$

<u>eg</u>

winding # around P is -1

<u>Note</u> given P, and loops C_1 and C_2 with same endpoint, then winding # of $C_1 + C_2$ around P is winding # of C_1 + winding # of C_2

Green's thm for multiply connected regions

$$\int_{\text{red region}} \text{curl } \vec{F} \, dx\,dy = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$$

<u>critical</u> C_3 is CW, C_1 & C_2 CCW

If curl $\vec{F} = 1$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\text{int}(C)} 1 \, dA = \text{area}(\text{int}(C))$$

<u>Surface Integrals</u>

<u>Recall</u>: 2 types of line integrals

① line integral of a scalar fcn (eg on $\mathbb{R}^2$)

For f def'd on a domain containing a curve C, we can take

$$\int_C f \, ds \qquad ds = \text{arclength}$$

<u>Note</u> ds always positive, and reversing the orientation of C doesn't change $\int_C f \, ds$

② line integral of a vector fcn.

For $\vec{F}$ def'd on a domain containing C, we take

$$\int \vec{F} \cdot d\vec{r}$$

Q/ Why dot product?

A/ Bc we have 2 vectors

$\vec{F}(x_i, y_i)$ and $\Delta x_i \hat{i} + \Delta y_i \hat{j} = \Delta \vec{r}$

and we want a scalar $\Rightarrow$ dot prod

$\vec{F}(x_i, y_i)$ and $\Delta x_i \hat{i} + \Delta y_i \hat{j} = \Delta \vec{r}$

and we want a scalar $\Rightarrow$ dot prod!

Note $d\vec{r}$ is a vector and reversing the orientation of C negates the integral

Recall defined using a Riemann sum, but computed using a parameterization.

Now surfaces

Let S be a bounded surface in $\mathbb{R}^3$.

ie, suppose we have fcn

$\vec{r}(s,t) = (x(s,t), y(s,t), z(s,z))$

$\vec{r}: D \to \mathbb{R}^3$

$D \subseteq \mathbb{R}^2$ take $D = [a,b] \times [c,d]$

Assume x, y, z have continuous partials $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$

Then the subset of $\mathbb{R}^3$ traced out by $\vec{r}(s,t)$ for $(s,t) \in [a,b] \times [c,d] \subseteq \mathbb{R}^2$ is the kind of surface we care about.

① for surface Let f be a scalar fcn defined on some domain in $\mathbb{R}^3$ containing a surface S.

will define $\int_S f\,dA$ $A = $ Area

Riemann sum. Break S into N little surfaces S_i

For $i = 1, 2, \ldots, N$

- Choose $(x_i, y_i, z_i) \in S_i$ $\forall_i$
- Then consider $\sum_{i=1}^{N} f(x_i, y_i, z_i)\,\text{area}(S_i)$
- define $\int_S f\,dA$ to be $\lim_{\text{mesh}\to 0}$ where $\text{mesh} = \max_i(\text{diam}(S_i))$

To compute

- Convert dA to $ds\,dt$
- Given a little rectangle w/ sides Δs & Δt it maps via $\vec{r}$ to a little parallelogram in S with sides $\frac{\partial \vec{r}}{\partial s} \cdot \Delta s$ and $\frac{\partial \vec{r}}{\partial t} \Delta t$

Area of a parallelogram:

$$\left\| \frac{\partial \vec{r}}{\partial s} \Delta s \times \frac{\partial \vec{r}}{\partial t} \Delta t \right\| = \Delta A$$

$$= \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \Delta s \Delta t$$

$$\Rightarrow \int_S f\,dA = \iint_{[a,b]\times[c,d]} f \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| ds\,dt$$

Q/ What is $\frac{\partial \vec{r}}{\partial s}$?

A/ A VVF of s, t that outputs vectors in $\mathbb{R}^3$

② For surfaces

Suppose we have $\vec{F}(x, y, z) = P\hat{i} + Q\hat{j} + R\hat{k}$

Will consider

$$\int \underbrace{\left(\vec{F}, \frac{\partial \vec{r}}{\partial s}, \frac{\partial \vec{r}}{\partial t} \right)}_{\text{need this to be a scalar}} ds\,dt$$

↳ det ?

$$\begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

In book consider $\vec{F} \cdot \hat{n}$

↳ this is the same bc $\vec{n} = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$

and $\det \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = V_1 \cdot (V_2 \times V_3)$